MATH 20D: Differential Equations Spring 2023
Homework 4

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Remember to list the sources you used when completing the assignment.
Below *NSS* is used to reference the text *Fundamentals of Differential Equations (9th edition) by Nagle, Saff, Snider*

Question (1)**.** *(a) Calculate the Wronskian for the pairs of functions listed below. In each case, determine whether the functions are linearly dependent or linearly independent on the domain* $\mathbb{R}$.

(i) $y_1(t) = \frac{1}{2}(e^t + e^{-t}), \quad y_2(t) = \frac{1}{2}(e^t - e^{-t})$ ***(ii)*** $y_1(t) = t^2, \quad y_2(t) = 2t|t|.$

(iii) $y_1(t) = e^{2t}, \quad y_2(t) = te^{2t}$ ***(iv)*** $y_1(t) = \sin(t), \quad y_2(t) = e^t\cos(t).$

(b) Determine which of the pairs considered in parts (i)-(iv) could occur as a pair of solution to a differential equation of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

where $p(t)$ *and* $q(t)$ *are continuous functions defined on* $(-\infty, \infty)$.

Question (2)**.** ***(NSS 4.7.32)*** *Let* $a, b \in \mathbb{R}$ *and suppose* $p(t)$ *and* $q(t)$ *are continuous functions defined on the interval* (a, b). *Let* $y_1(t)$ *and* $y_2(t)$ *be solutions to the equation*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

and write $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ *for the Wronskian of the functions* y_1 *and* y_2.
(a) Show that the Wronskian W *satisfies the first order differential equation*

$$W'(t) + p(t)W(t) = 0$$

(b) Let $t_0 \in (a, b)$. *By solving the differential equation in (a) show that*

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} p(\tau)d\tau\right) \qquad \textit{(Abel's Formula)}$$

(c) Using the result of (b), show that if there exists a single $t_0 \in (a, b)$ *such that*

$$W(t_0) = 0$$

then $W(t) = 0$ *for all* $t \in (a, b)$.

(d) *(Optional) Show that if $W(t) \equiv 0$ then y_1 and y_2 are linearly dependent.* Hint: I would suggest reading the proof of Lemma 1 in NSS §4.2

Question (3). ***(NSS 4.7.34)*** *All that is known concerning a mysterious differential equation*

$$y'' + p(t)y'(t) + q(t)y(t) = g(t) \tag{0.1}$$

is that the functions t, t^2, and t^3 are solutions.

(a) *Determine two linearly independent solutions to the corresponding homogeneous differential equation*

$$y'' + p(t)y'(t) + q(t)y(t) = 0. \tag{0.2}$$

Hint: By the result discussed in lecture 11, we can write $t = y_p(t) + y_c(t)$ and $t^2 = y_p(t) + u_c(t)$ where $y_p(t)$ is a particular solution to (0.1) and the functions $y_c(t)$ and $u_c(t)$ are solutions to (0.2). What is the function $y_c(t) - u_c(t)$? Why does it solve (0.2)?

(b) *Find a solution to the equation (0.1) satisfying $y(2) = 2$ and $y'(2) = 5$.*

(c) *Determine the function $p(t)$?* Hint: Use Abel's formula from Problem 2. Your solution to (a) will also be useful.

Question (4). *Using the method of variation of parameters, give general solutions to the following differential equations*

(a) $y'' + y = \sec(t)$, ***(b)*** $y'' - 2y' + y = t^{-1}e^t$, ***(c)*** $y'' + 16y = \sec(4t)$,

(d) $y'' + 4y' + 4y = e^{-2t}\log(t)$, ***(e)*** $y'' + y = \sec^3(t)$ ***(f)*** $y'' - 6y' + 9y = t^{-3}e^{3t}$.

Question (5). ***(NSS 4.6.22,23,24)*** *In the problems below use variations of parameters to find a general solution to the differential equation given that the functions y_1 and y_2 are linearly independent solutions to the corresponding homogeneous equation for $t > 0$.*

(a) $t^2y'' - 4ty' + 6y = t^3 + 1$; $y_1 = t^2$, $y_2 = t^3$.

(b) $ty'' - (t+1)y' + y = t^2$; $y_1 = e^t$, $y_2 = t + 1$.

(c) $ty'' + (1-2t)y' + (t-1)y = te^t$; $y_1 = e^t$, $y_2 = e^t\log(t)$.

Question (6). ***(NSS 4.7.41,42)*** *In each of the parts below, a differential equation and a non-trivial solution f are given. Find a second linearly independent solution using the method of reduction of order.*

(a) $t^2y'' - 2ty' - 4y = 0$, $t > 0$; $f(t) = t^{-1}$.

(b) $t^2y'' + 6ty' + 6y = 0$, $t > 0$; $f(t) = t^{-2}$.

(c) $t^2y'' + ty' + (t^2 - 1/4)y = 0$, $t > 0$; $f(t) = t^{-1/2}\cos(t)$.

Question (7). *Given that $f(t) = e^t$ is a solution to the homogeneous equation*

$$ty'' - (t+1)y' + y = 0, \qquad t > 0,$$

solve the initial value problem

$$ty'' - (t+1)y' + y = t^2e^t, \qquad y(1) = 2e^2, \qquad y'(1) = 2e^2.$$

Question (8). *Given that $f(t) = t$ is a solution to the homogeneous equation*

$$(1-t)y'' + ty' - y = 0 \qquad t > 0,$$

solve the initial value problem

$$(1-t)y'' + ty' - y = (1-t)^2, \qquad y(2) = 3 + e^2, \qquad y'(2) = 3 + e^2.$$